Sparsity-promoting Optimal Control of Spatially Invariant Systems

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Outline

❶ PROBLEM FORMULATION

- ⋆ **Sparsity promoting H_2 control**
- ⋆ **Efficient computation for spatially invariant systems**

❷ OPTIMIZATION ALGORITHM

- ⋆ **Alternating direction method of multipliers**
- ⋆ **Structured optimal control problem**

❸ EXAMPLES AND SCALING

- ⋆ **Linear computational complexity in the number of subsystems**

State-feedback H_2 problem

$$\dot{x} = Ax + B_1 d + B_2 u$$
$$u = -Fx$$

- CLOSED-LOOP H_2 NORM

$$J(F) := \lim_{t \to \infty} \mathcal{E}\left(x^T(t) Q x(t) + u^T(t) R u(t)\right)$$

- OPTIMAL STATE FEEDBACK GAIN

$$A^T P + PA + Q - PB_2 R^{-1} B_2^T P = 0$$
$$F = R^{-1} B_2^T P$$

⋆ F **is in general a dense matrix**

Sparsity-promoting H_2 problem

- ADD SPARSITY REQUIREMENTS

$$\text{minimize} \quad J(F) \quad + \quad \gamma\, \mathbf{card}\,(F)$$

$J(F)$ $\downarrow$ **optimize performance**

$\gamma\, \mathbf{card}\,(F)$ $\downarrow$ **promote sparsity**

Lin, Fardad, Jovanović, IEEE TAC '13

- CHALLENGES

  ⋆ **non-convex objective function**

  ⋆ **computationally complex for large systems**

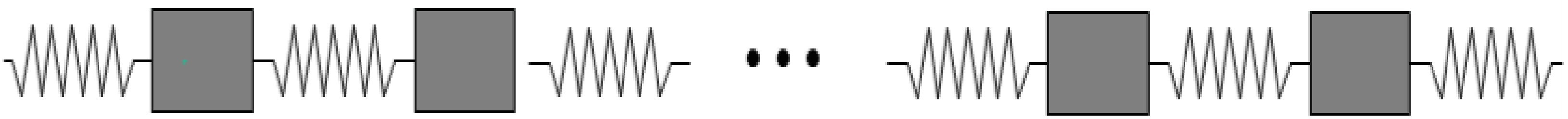
$$\mathcal{O}(N^3 m^3)$$

  ∗ N - number of subsystems
  ∗ m - number of states per subsystem

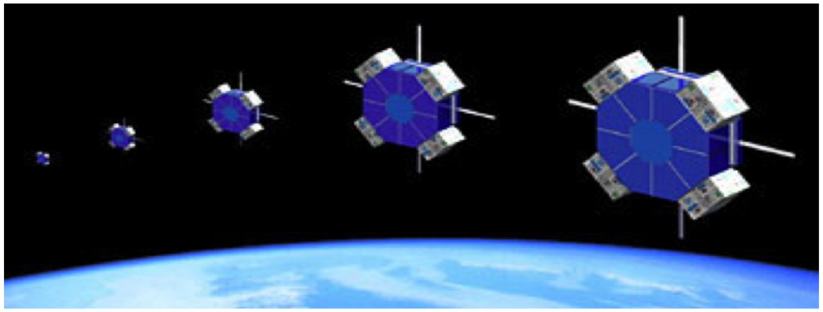
Spatially-invariant systems

- TRADEOFF GENERALITY FOR PERFORMANCE
  - ⋆ **restrict method to spatially-invariant systems**
  - ⋆ **develop more efficient method**

- EXAMPLES
  - ⋆ **infinite mass-spring system**



  - ⋆ **satellite system**



Spatially-invariant systems

- Apply discrete Fourier transform

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_1\hat{d} + \hat{B}_2\hat{u} \\ \hat{u} &= -\hat{F}\hat{x} \end{aligned}$$

$$\hat{F} = \Phi \, F \, \Phi^*$$

- Decouple H_2 norm

$$J(\hat{F}) = \sum_{i\,=\,1}^{N} J_i(\hat{F}_i) = \sum_{i\,=\,1}^{N} \text{trace}\left(\hat{P}_i\,\hat{B}_{1i}\,\hat{B}_{1i}^T\right)$$

⋆ **computational complexity of solving Riccati equation**

$$\mathcal{O}(N m^3)$$

⋆ **linear in the number of subsystems** N

Bamieh, Paganini, Dahleh, IEEE TAC '02

Sparsity-promoting H_2 problem

- ADDITIONAL REQUIREMENTS ON GAIN

  ⋆ **promote sparsity**

- OPTIMIZATION PROBLEM

$$\text{minimize} \quad \underbrace{J(\hat{F})}_{\text{Fourier domain: optimize performance}} \quad + \quad \underbrace{\gamma \sum_{i,j} W_{ij} \, |F_{ij}|}_{\text{physical domain: promote sparsity}}$$

Fourier domain: optimize performance

physical domain: promote sparsity

$g(F) \to$ convex relaxation of $\mathbf{card}(F)$

Alternating direction method of multipliers

- INTRODUCE CONSTRAINT

$$\begin{array}{ll} \text{minimize} & J(\hat{F}) \;+\; \gamma\, g(F) \\ \text{subject to} & \hat{F} \;-\; \Phi\, F\, \Phi^* \;=\; 0 \end{array}$$

⋆ **exploit structure in both domains**

- FORM AUGMENTED LAGRANGIAN

$$\mathcal{L}_\rho(\hat{F}, F, \Lambda) \;=\; J(\hat{F}) \;+\; \gamma g(F) \;+\; \mathrm{trace}\,(\Lambda^T(\hat{F} - \Phi F \Phi^*)) \;+\; \frac{\rho}{2}\,\|\hat{F} - \Phi F \Phi^*\|_F^2$$

Alternating direction method of multipliers

- Solve optimization problem using sequence of iterations (ADMM):

1. **Optimize performance**

$$\hat{F}^{k+1} := \arg\min_{\hat{F}} \mathcal{L}_\rho(\hat{F}, F^k, \Lambda^k)$$

  ⋆ **in Fourier domain**

2. **Promote sparsity**

$$F^{k+1} := \arg\min_{F} \mathcal{L}_\rho(\hat{F}^{k+1}, F, \Lambda^k)$$

  ⋆ **in physical domain**

3. **Update Lagrange multiplier**

$$\Lambda^{k+1} := \Lambda^k + \rho(\hat{F}^{k+1} - \Phi F^{k+1} \Phi^*)$$

Performance optimization step

$$\text{minimize} \quad J(\hat{F}) + (\rho/2)\|\hat{F} - \hat{U}^k\|_F^2$$

$$\hat{U}^k := \Phi F^k \Phi^* - (1/\rho)\Lambda^k$$

- DECOUPLE PROBLEM

$$\text{minimize} \quad \sum_{i=1}^{N} \left(J_i(\hat{F}_i) + (\rho/2)(\hat{F}_i - \hat{U}_i^k)^2 \right)$$

- EFFICIENCY GAIN

  ⋆ **solve Lyapunov equations for each block**

Lin, Fardad, Jovanović, IEEE TAC '13

$$\mathcal{O}(N m^3)$$

  ⋆ **add subsystem $\rightarrow$ solve one more set of Lyapunov equations**

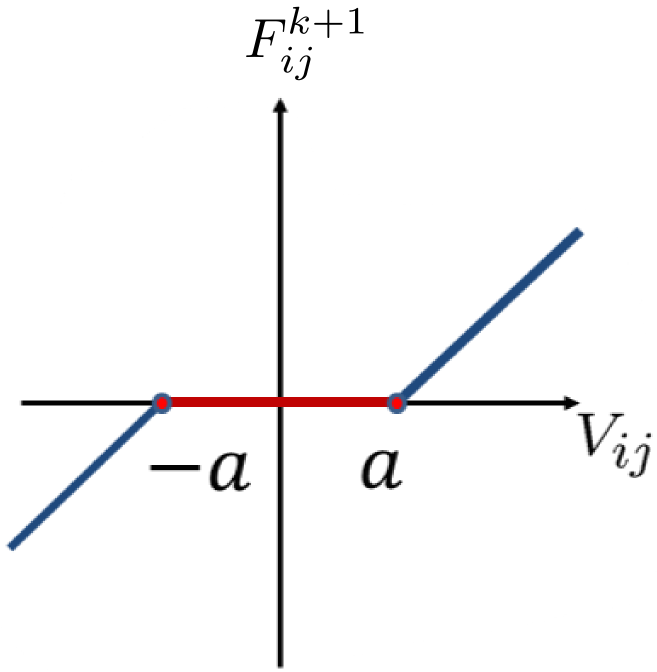
Sparsity-promoting step

$$\text{minimize} \quad \gamma g(F) \; + \; (\rho/2)\|F - V^k\|_F^2$$

$$V^k \; := \; \hat{F}^{k+1} \; + \; (1/\rho)\Lambda^k$$

- SOLUTION: SOFT-THRESHOLDING OPERATOR

$$F_{ij}^{k+1} = \begin{cases} (1 - a/\|V_{ij}\|_F)V_{ij}, & \|V_{ij}\|_F > a \\ 0, & \|V_{ij}\|_F \leq a \end{cases}$$



⋆ $a = (\gamma/\rho)W_{ij}$

Structured optimal control problem

- OPTIMIZE IDENTIFIED STRUCTURE

$$\begin{array}{ll} \text{minimize} & J(\hat{F}) \\ \text{subject to} & \hat{F} - \Phi F \Phi^* = 0, \quad F \in \mathcal{S} \end{array}$$

$\mathcal{S} \to$ subspace of sparsity patterns identified via ADMM

- REWRITE WITH INDICATOR FUNCTION ϕ

$$\begin{array}{ll} \text{minimize} & J(\hat{F}) + \phi(F) \\ \text{subject to} & \hat{F} - \Phi F \Phi^* = 0 \end{array}$$

Solving structured optimal control problem using ADMM

- PERFORMANCE OPTIMIZATION STEP

- STRUCTURE ENFORCEMENT STEP

$$\begin{array}{ll} \text{minimize} & \frac{\rho}{2}\|F - V^k\|_F^2 \\ \text{subject to} & F \in \mathcal{S} \end{array}$$

$$V^k = \hat{F}^{k+1} + (1/\rho)\Lambda^k$$

⋆ **Solution**

$$F^{k+1} = V^k \circ I_{\mathcal{S}}$$

$I_{\mathcal{S}}$ - **structural identity**

$$F = \begin{bmatrix} * & * & & \\ * & * & * & \\ & * & * & * \\ & & * & * \end{bmatrix} \quad \Rightarrow \quad I_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & & \\ 1 & 1 & 1 & \\ & 1 & 1 & 1 \\ & & 1 & 1 \end{bmatrix}$$

EXAMPLES

First-order system with $N = 5$ nodes distributed over a circle

- SYSTEM DYNAMICS:

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$$B_1 = B_2 = Q = R = I$$

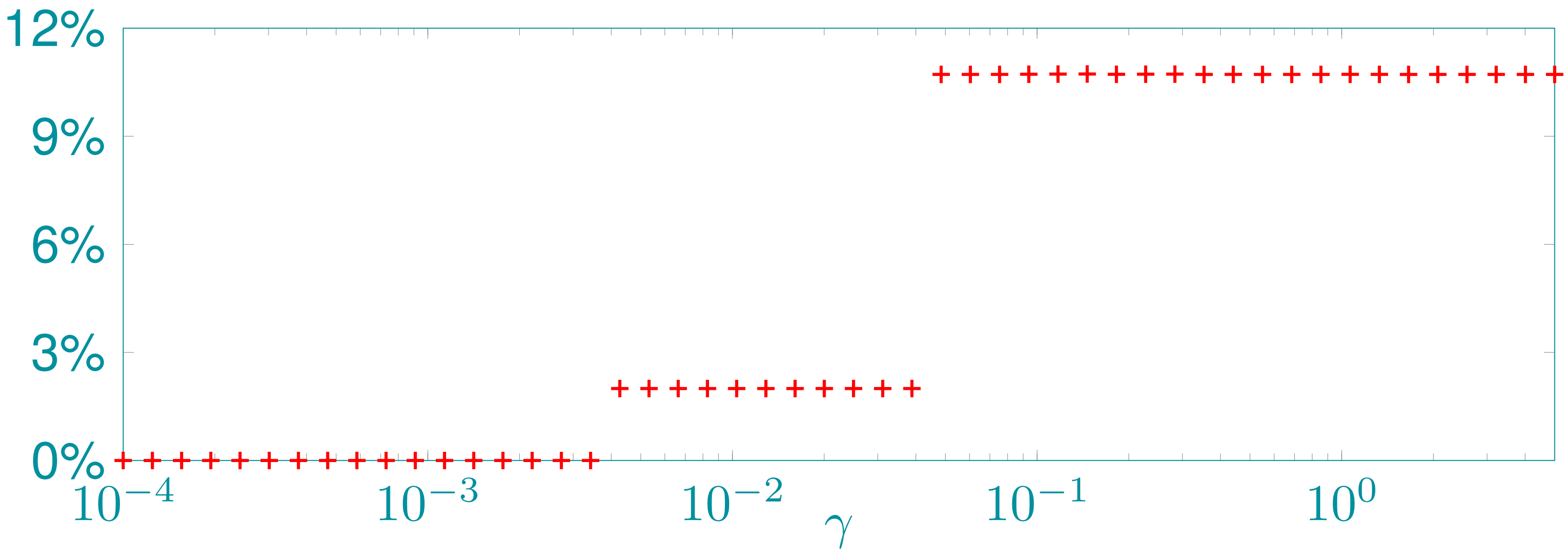
- CENTRALIZED GAIN

$$F_c = \begin{bmatrix} 0.3838 & 0.1961 & 0.1120 & 0.1120 & 0.1961 \\ 0.1961 & 0.3838 & 0.1961 & 0.1120 & 0.1120 \\ 0.1120 & 0.1961 & 0.3838 & 0.1961 & 0.1120 \\ 0.1120 & 0.1120 & 0.1961 & 0.3838 & 0.1961 \\ 0.1961 & 0.1120 & 0.1120 & 0.1961 & 0.3838 \end{bmatrix}$$

Sparse and centralized controllers comparison

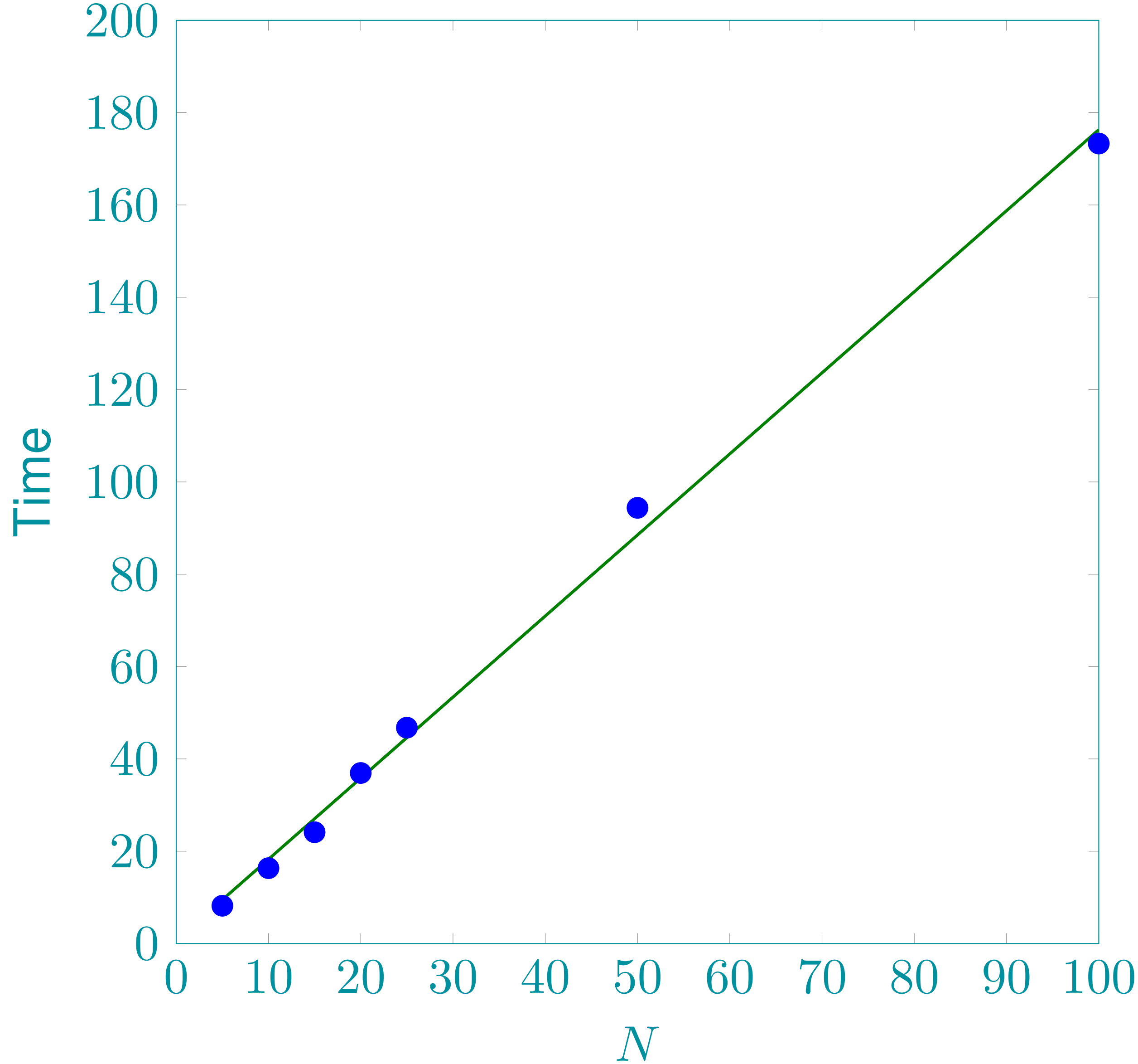
- PERFORMANCE DEGRADATION

$$(J(F) - J(F_c))/J(F_c)$$



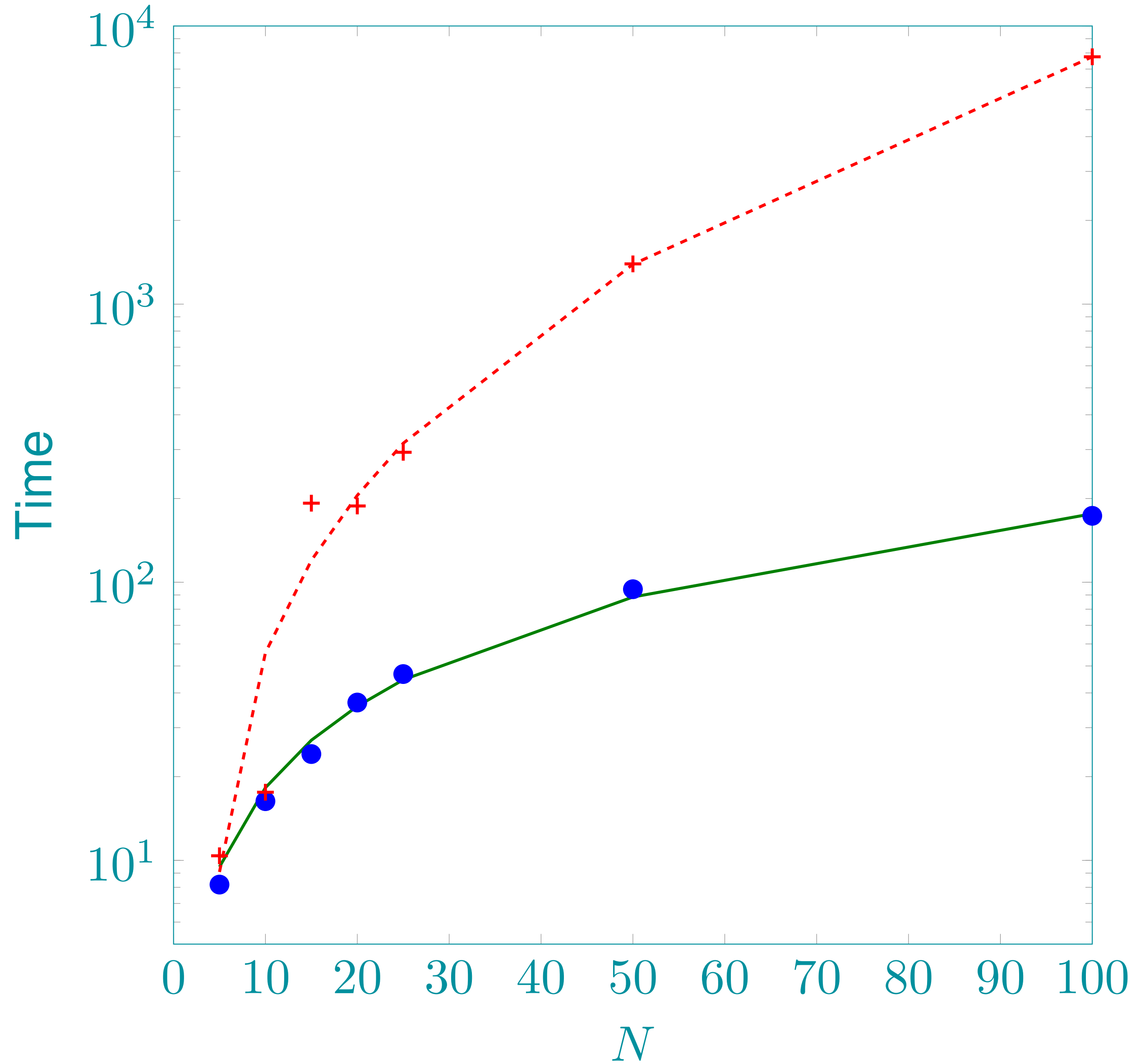
Computational complexity

- LINEAR SCALING IN THE NUMBER OF SUBSYSTEMS N



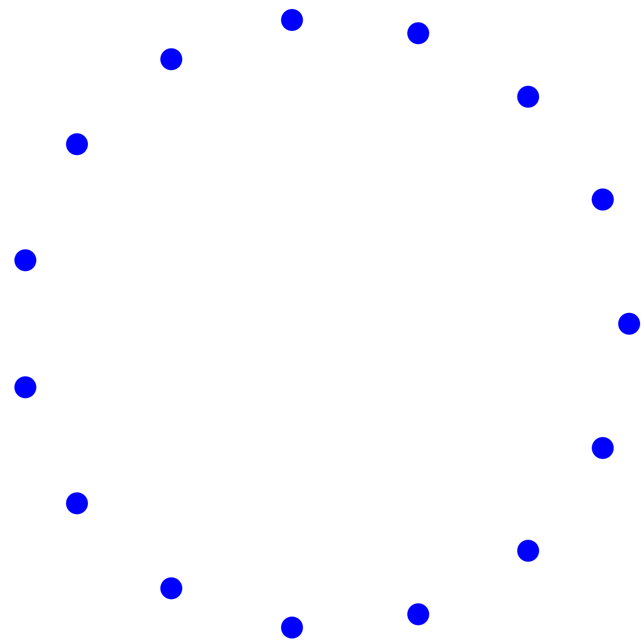
Computational Complexity

- ALGORITHM (●) EXPLOITS VS. (+) DOES NOT EXPLOIT CIRCULANT STRUCTURE
  - ⋆ **(●) linear vs. (+) cubic scaling in the number of subsystems**



Distributed system with $N = 15$ subsystems

- ⋆ EVENLY DISTRIBUTED ON A CIRCLE

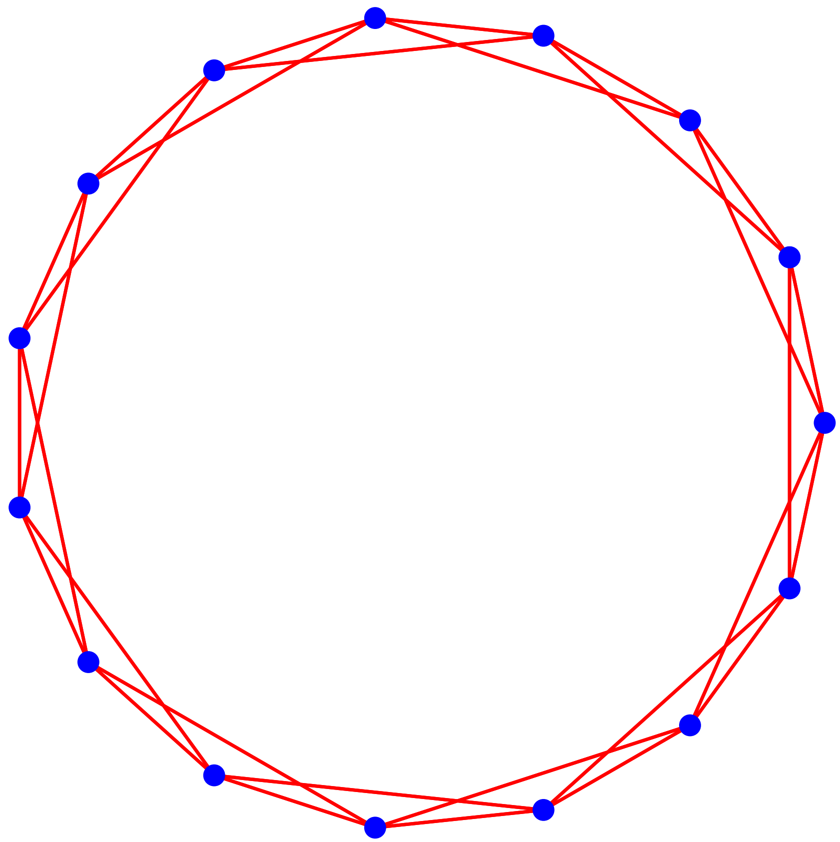


$$\begin{bmatrix} \dot{p}_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} p_i \\ v_i \end{bmatrix} + \sum_{j \neq i} e^{\alpha(i,j)} \begin{bmatrix} p_j \\ v_j \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (d_i + u_i)$$

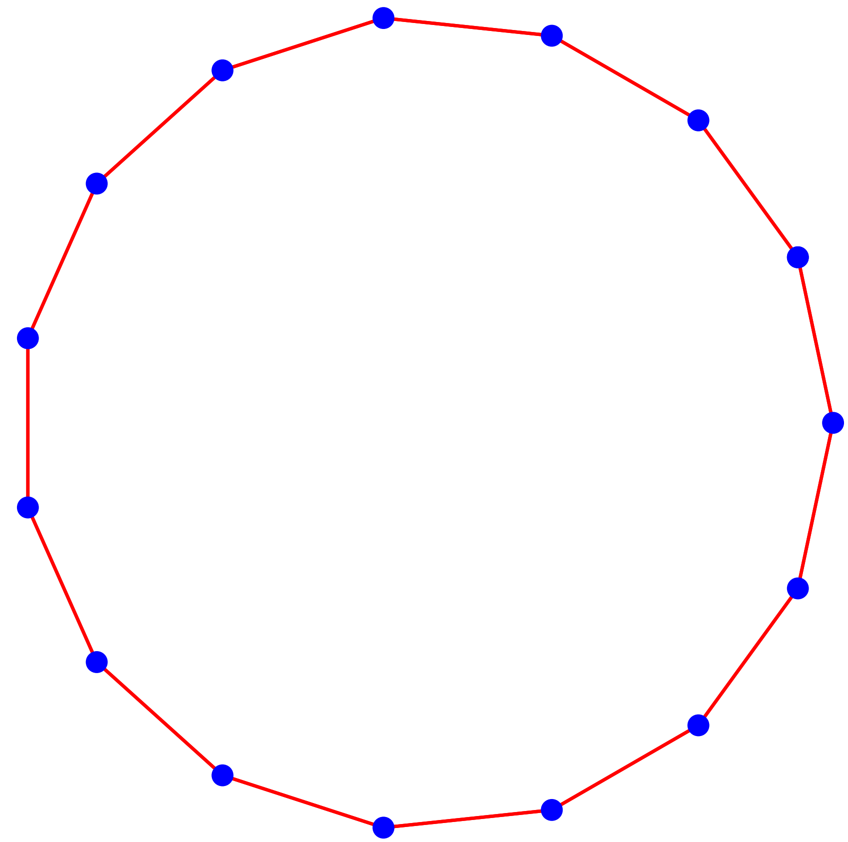
- ⋆ $\alpha(i,j)$: EUCLIDEAN DISTANCE BETWEEN SUBSYSTEMS i AND j

Identified communication graphs

$\gamma = 0.1389$

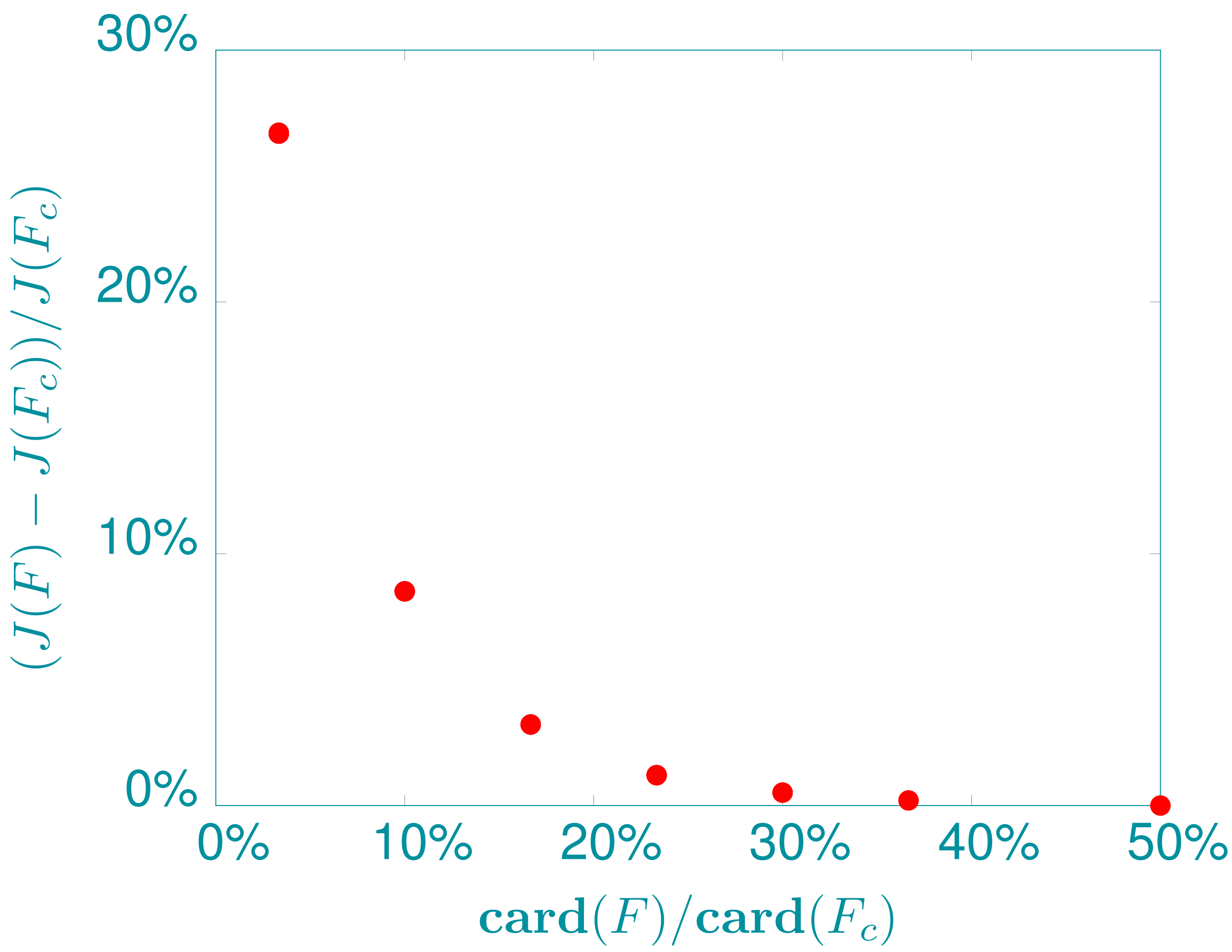


$\gamma = 0.9103$



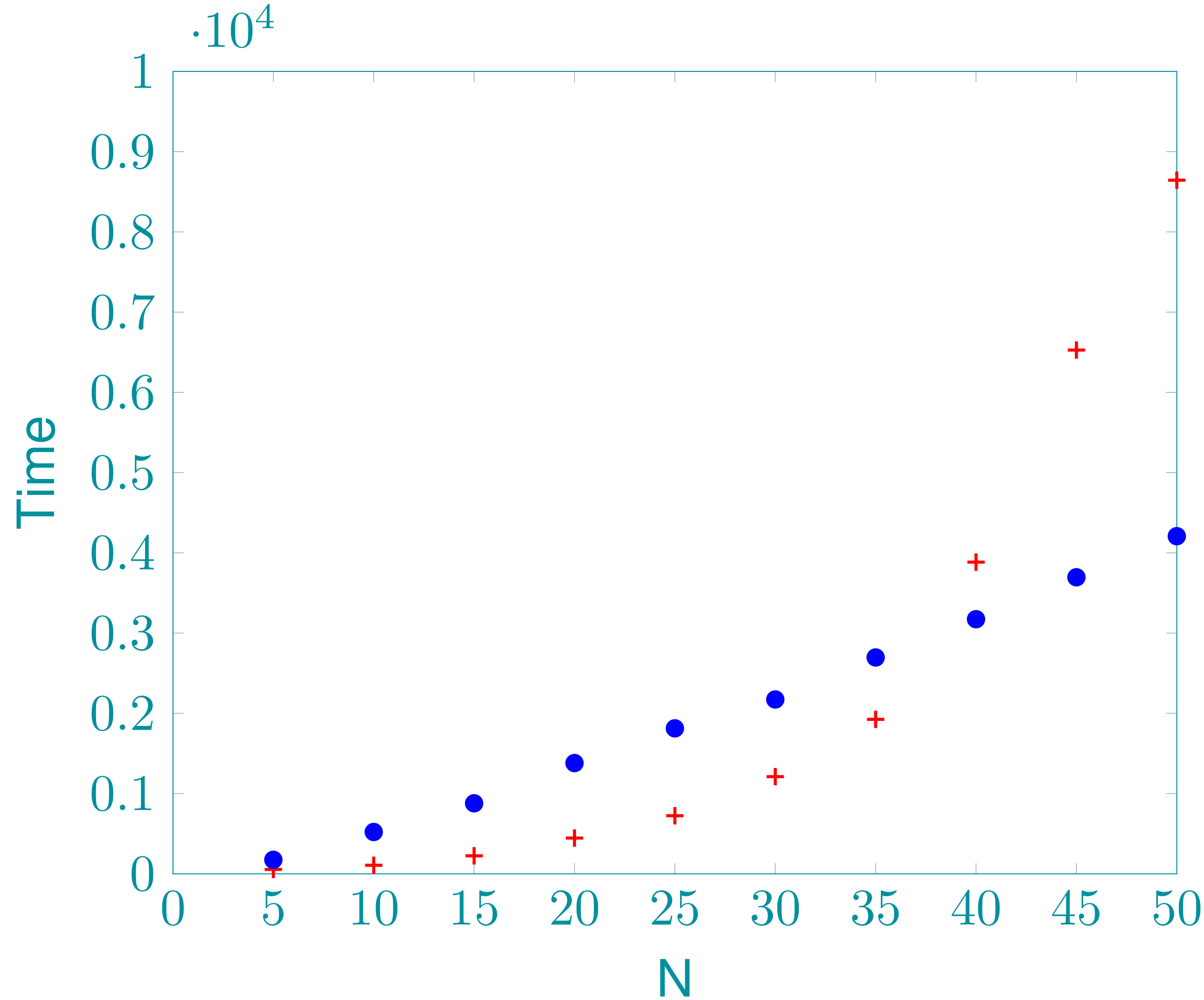
Sparsity vs. Performance Degradation

- COMPARED TO CENTRALIZED CONTROLLER



Computational Complexity

- Algorithm (•) exploits vs. (+) does not exploit circulant structure
  - ⋆ **(•) linear vs. (+) cubic scaling in the number of subsystems**



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